

The pendulum clock: a venerable dynamical system

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Abstract

The steady-state motion of a weight-driven pendulum clock is shown to be a stable limit cycle. An explicit solution is obtained via Green functions. The pendulum amplitude is found to be a simple function of parameters. The key role played by the anchor escapement is discussed and placed in historical context.

1. Introduction

A longcase or weight-driven pendulum clock is a dynamical system consisting of a damped pendulum oscillator acting under a nonlinear force. The frequency of the forcing term is that of the damped pendulum, and so this system is an example of self-excited oscillation [1]. Potential energy of the weight is converted into kinetic energy of the pendulum via the clock *escapement* mechanism.

The physics of this system is instructive, since the student learns about damped harmonic motion, self-excited oscillation, stability and limit cycles, all from a familiar and historically important device. The history of clock development is very interesting; in section 2 we summarize relevant aspects of this, to place in context the development of the *anchor escapement*, which we analyse below.

The first mathematical analysis of pendulum clocks was that of Airy in 1826. He became interested in timekeeping through his astronomical work. (Airy also advocated a national time standard for Britain.) His analysis is outlined in section 3.

Longcase clocks are tall, because of the pendulum and the weight. Known widely as ‘coffin’ clocks, due to their shape and opening front, they have been known in England since 1876 as ‘grandfather’ clocks [2]. The pendulum is about 1 m long, to give a period of 2 s. The anchor engages the escape wheel twice per cycle (see the caption to figure 1), i.e. once per second. This generates the clock’s ‘tick’. Thus, the escapement mechanism does more than just maintain the pendulum oscillation: it regulates the period. The escapement quite literally makes the clock tick. The pendulum oscillation is maintained at the natural frequency and is stable against minor disturbances. We shall demonstrate in this paper that the motion

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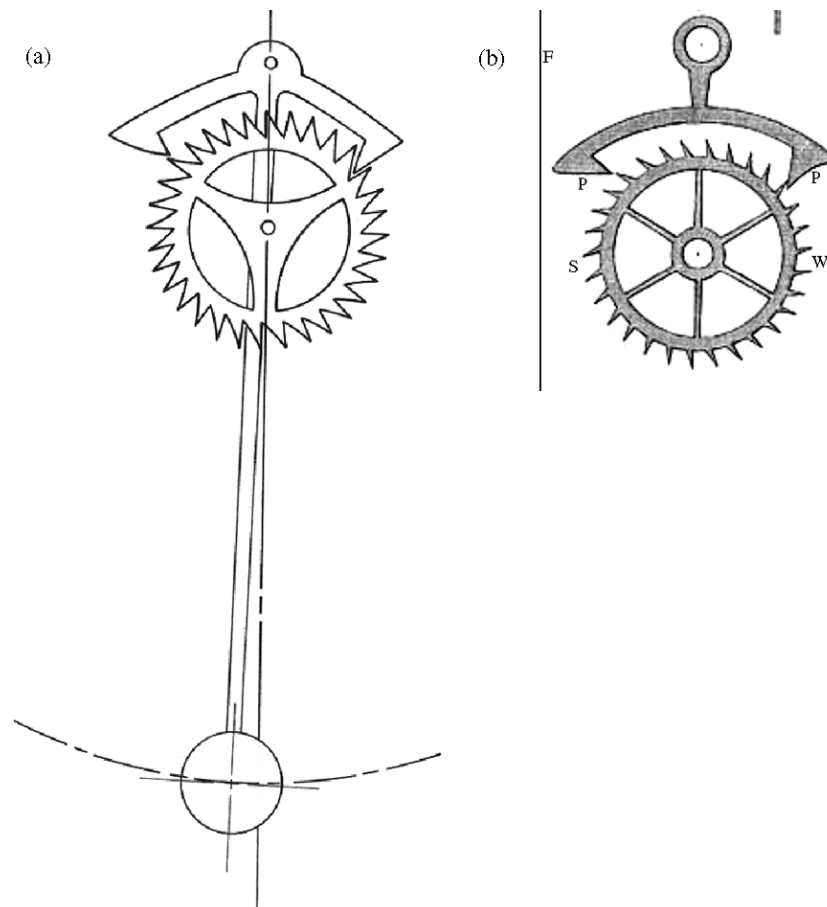


Figure 1. (a) The anchor escape is fixed to the pendulum, and oscillates with it. The toothed escape wheel is attached via a gear train to a spindle, around which a string is wrapped and attached to a weight (not shown), so that the spindle and the escape wheel will rotate clockwise as the weight descends. This motion is interrupted by the two teeth of the anchor engaging the escape wheel. The anchor shape is such that, when one tooth disengages the escape wheel, the second tooth engages it a very short time later, so that the escape wheel rotates through a small angle (and the clock hand to which it is attached rotates through $1/60$ of a circle). This action occurs twice per pendulum cycle. The effect of the escape wheel upon the anchor gives a small impulse to the pendulum motion, and produces the characteristic tick-tock sound. Note that this diagram is greatly simplified, for clarity. (b) Detail of the anchor and escape wheel, from an 1832 encyclopaedia.

of a pendulum, regulated by an anchor escapement, is a stable limit cycle, and shall derive an explicit solution using Green functions (sections 4 and 5). The analysis is confirmed by numerical results (section 6). We finish with a brief discussion on energy loss.

2. Historical perspective

Application of the pendulum to mechanical timekeeping traditionally dates to 1583. In this year, the young Galileo postulated the *isochronous* (constant period, independent of amplitude) nature of a lamp swinging in Pisa cathedral. In 1641, when old and blind, he is supposed to have described to his son Vincenzo how a clock based upon the pendulum could be built [3].

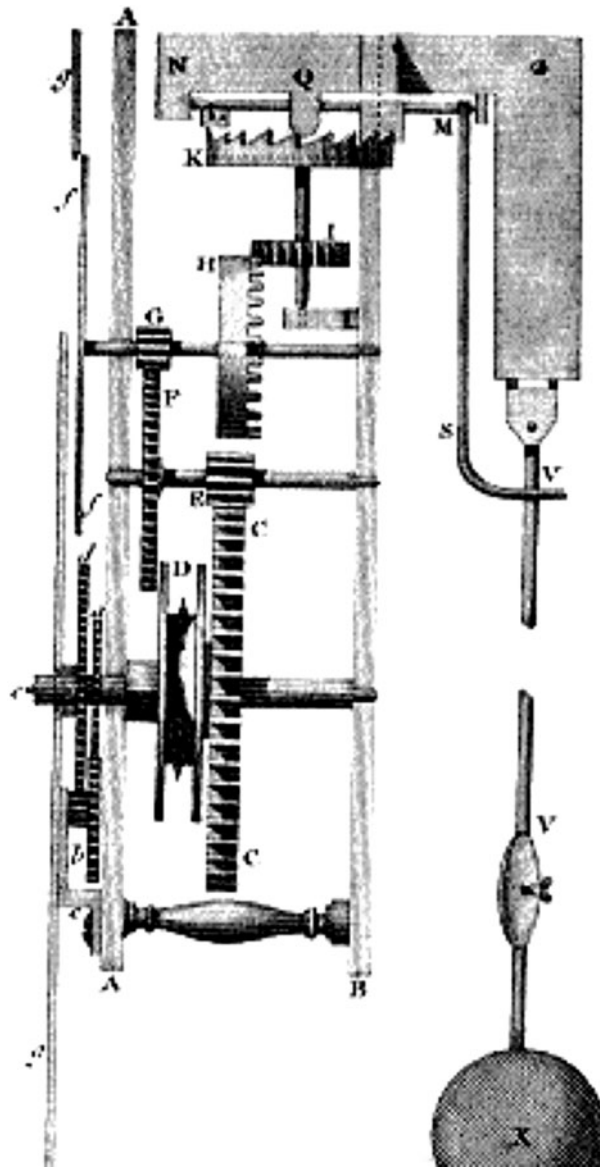


Figure 2. Huygen's clock, viewed from the side. The clock face at left is connected by a gear train to the crown escapement K.

Indeed, one may have been constructed, but all records were destroyed by Vincenzo when delirious with fever in 1649. The first certain claim for a pendulum clock is that designed by Huygens, and built by Coster in 1656. Seven such Coster clocks survive.

Huygens interest in timekeeping had led him to show theoretically that the cycloid was a *tautochrone* under uniform gravity. That is, a body following a cycloid curve will descend to the bottom of the curve in a fixed time, independent of initial position. He showed how to modify a pendulum motion so that it had this property for all amplitudes, and so was isochronous for all amplitudes, and not just in the limit of small amplitudes observed by Galileo [4]. Huygens

was interested in solving the longitude problem then troubling navigators [3, 5] and proposed to solve it with an accurate pendulum clock².

He developed a recoil escapement to regulate the pendulum and prevent it from running down (shown in figure 2), conducted trials of his clocks in 1662 and 1686, and patented his ideas in 1664–5. His clocks were not accurate at sea, due to ship pitching and rolling motion interfering with the pendulum oscillation. It would take another century before a marine chronometer of sufficient accuracy to provide useful longitude estimation was developed (by Harrison, in 1761 [5]).

Huygen's clocks were accurate on land to within a minute a day. Later models improved this to 10 s a day. The next significant improvement came about by marrying the pendulum clock to a new anchor escapement developed in England [2]. Within weeks of Huygen's patents being granted, the production rights were secured by Ahasuerus Fromanteel, and this ushered in the age of English longcase clocks, which dominated horology for a century. The French, spurred also by the longitude problem, dominated clock and watchmaking over the period 1770–1840, in turn stimulating the Swiss clock industry.

Accuracy of pendulum clocks was improved by Graham in 1721 to 1 s/day. This required an improved escapement [3, 6] and a method for compensating the change in pendulum length with ambient temperature. (It is straightforward to show that a steel pendulum will lose 1 s/day if the temperature is increased by about 2 °C.) Later refinements include a multitude of new escapement devices, a polished pendulum oscillating in a partial vacuum, and a fused-quartz pendulum, the length of which does not vary significantly with temperature. Graham made two month clocks for Halley, which were in use until the beginning of the 20th century, and which '... are still keeping time in the Royal Observatory to within a few seconds a week' [7]. Through these refinements and others, mechanical pendulum clocks remained the most accurate timekeeping machines until around 1930, by which date their accuracy was a few milliseconds per day.

The power source of a grandfather clock is a weight attached to a spindle, which slowly lowers the weight, transferring the potential energy into pendulum kinetic energy. The escapement transfers just enough energy during each pendulum cycle to compensate for loss of energy due to friction. By this method, the pendulum will oscillate for a week, typically, before the weight needs to be returned to its original height, as the clock is wound up. Without this input of energy, the pendulum will oscillate for a few hours only. Hooke claims to have invented the anchor escapement [3, 6]. It was first applied to pendulum clocks by William Clement in 1671. It has an advantage over earlier mechanisms in that it interferes less with the pendulum swing. (The older *crown wheel* or *verge* escapement, shown labelled K in figure 2, had been employed for over 400 years. It required the pendulum to oscillate at large amplitudes, so that it was not isochronous. This deviation of the pendulum oscillation from the isochronous cycloid is called *circular error* by horologists [5].) Thus, maintaining pendulum motion required less power with the newer mechanism. The name arises from their shape, as shown in figure 1. Different escapements have different actions and efficiencies. They wear at different rates. Some require lubrication, and some do not. Some are robust, whereas other designs are fragile³.

² The problem of accurately determining longitude prompted firstly the Spanish, and later the Dutch, French and British, to offer large rewards to the inventor of a satisfactory method. This provided a great impetus to the development of timekeeping, and occupied the minds of Galileo, Pascal, Hooke, Huygens, Leibniz and Newton, amongst others. Because of the prize at stake, and perhaps also the prestige, claims over priority and patents were common and acrimonious at this time, particularly those involving Huygens and Hooke [3].

³ There are a number of interesting websites containing illustrations of different escapement mechanisms, some of which are animated. This animation greatly assists in understanding the mechanism, which in some cases (such as Harrison's *grasshopper escapement*) is quite complicated. We refer the interested reader to, for example, <http://home.talkcity.com/Terminus/mvhw/escapement.html>. There is a qualitative description of pendulum clocks, which includes animations, at <http://www.howstuffworks.com/clock.htm>. A more detailed simulation can be found at <http://www.materialworlds.com/sims/PendulumClock/install.html>.

3. Airy's analysis

This paper [8] is an early analysis of clocks and watches (read to the Cambridge Philosophical Society in 1826, though published four years later) by an eminent physicist, who was unaware of our modern ideas concerning nonlinear systems. Here we shall outline the assumptions and the rather neat calculations. We will omit the detailed applications to particular cases, and shall leave to the interested reader Airy's conclusions concerning best escapement design. (He recommends a type of pinwheel escapement.) Our main interest in this work is the contrast of Airy's 'classical' analysis with our modern approach given below, which is presented with the benefit of current knowledge of limit cycles and other aspects of nonlinear dynamics.

The linearized pendulum without friction is described by the harmonic oscillator equation

$$\ddot{\theta} \approx -\omega_0^2 \theta, \quad \omega_0^2 = \frac{g}{\ell} \quad (1)$$

where θ is angular displacement of the pendulum from vertical, g is acceleration due to gravity and ℓ is pendulum length. We adopt dot notation for the time derivative. The solutions for angular displacement and velocity are familiar:

$$\theta = a \sin(\omega_0 t + \phi) \quad \dot{\theta} = \omega_0 a \cos(\omega_0 t + \phi). \quad (2)$$

Now suppose there is a small additional force f (due to friction, or the escapement mechanism, or to nonlinear pendulum terms), so that (1) becomes

$$\ddot{\theta} \approx -\omega_0^2 \theta + f. \quad (3)$$

Airy seeks solutions to this which retain the form of equation (2), but now with time-dependent amplitude a and phase ϕ . This leads to the following coupled equations:

$$\dot{a} = \frac{f}{\omega_0} \cos(\omega_0 t + \phi) \quad \dot{\phi} = -\frac{f}{\omega_0 a} \sin(\omega_0 t + \phi). \quad (4)$$

Airy expects f to be small, and so greatly simplifies equations (4) by assuming a, ϕ on the right-hand side of equations (4) to be constant, since the time dependence is of order f^2 . He points out that, for escapements, the dependence is really upon displacement θ rather than upon time (in modern parlance we say the system is *autonomous*), which leads to

$$\frac{da}{d\theta} = \frac{f}{\omega_0^2 a}, \quad \frac{d\phi}{d\theta} = -\frac{f}{\omega_0^2 a^2} \frac{\theta}{\sqrt{a^2 - \theta^2}}. \quad (5)$$

From these equations, Airy calculates the increase in amplitude, and the fractional increase in period τ , over one cycle:

$$\Delta a = \frac{1}{\omega_0^2 a} \int_{-\pi}^{\pi} d\theta f, \quad \frac{\Delta \tau}{\tau} = \frac{1}{2\pi \omega_0^2 a^2} \int_{-\pi}^{\pi} d\theta \frac{f \theta}{\sqrt{a^2 - \theta^2}}. \quad (6)$$

For the regular oscillations required of clocks and watches, we would like these two quantities to be as small as possible. Note from (6) that the period is constant if the force f is an even function of θ , whereas the amplitude is constant if f is an odd function. In general it is difficult to make both Δa and $\Delta \tau$ zero. Airy investigates many forms for f : circular error, different types of friction and different escapement action. He concludes, rightly, that the *dead-beat* escapement is best. This had been developed by Graham in 1721; earlier anchor escapement were of the *recoil* type, which retarded the pendulum swing during half the cycle (causing the clock hands to recoil), and boosted it during the other half cycle. The dead-beat mechanism avoided recoil, with consequent reduction in friction, and particularly in wear. We shall discuss the dynamics of these escapement variants below.

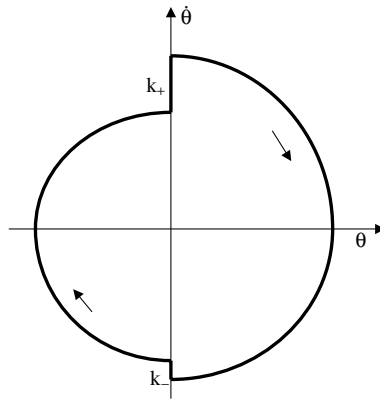


Figure 3. Phase diagram for a pendulum clock with recoil escapement. Without friction, the phase plot is an ellipse. Friction turns it into a spiral to the origin. Escapement restores the spiral to a limit cycle. The anchor mechanism contributes a small impulse twice a cycle, over a short time interval Δt , as the pendulum passes its lowest point. One impulse k_+ increases pendulum angular speed, whereas the second impulse k_- reduces it (causing recoil). If k_+ exceeds k_- then enough energy is being input to the pendulum to offset the effects of friction, and a limit cycle results.

4. Escapement action and the limit cycle

Consider again the linearized simple harmonic oscillator, this time with friction explicit:

$$\ddot{\theta} + b\dot{\theta} + \omega_0^2\theta \approx 0 \quad (7)$$

where b is friction coefficient. For small damping ($b \ll 2\omega_0$) the general solution is [9]

$$\theta(t) = \theta_0 \exp(-\frac{1}{2}bt) \cos(\omega t - \psi), \quad \omega^2 = \frac{g}{\ell} - \frac{1}{4}b^2. \quad (8)$$

Henceforth we set the constant phase to $\psi = 0$. Note that the frequency is reduced somewhat from the natural pendulum frequency ω_0 of equation (1), and that the phase portrait consists of a spiral into the origin. The escapement action changes this, as sketched in figure 3. The spiral is interrupted by small impulses k_{\pm} imparted to the pendulum at $\theta = 0$ due to the anchor engaging the escape wheel. These impulses add just enough energy to overcome the effects of damping, and a limit cycle results⁴. We shall now establish this, and show that the limit cycle is stable.

Consider the phase trajectory to be at $\theta = 0$ at times t_n , where $\omega t_n = 2n\pi$, where n is an integer. The period is thus $\tau = t_{n+1} - t_n$. From equation (8) we see that

$$|\dot{\theta}(t_{n+1})| = \exp(-\frac{1}{2}b\tau)|\dot{\theta}(t_n)| + k_+ - k_-. \quad (9a)$$

In a more convenient notation $x_n \equiv |\dot{\theta}(t_n)|$, $r = \exp(-\frac{1}{2}b\tau)$, $k \equiv k_+ - k_-$ equation (9a) is expressed as

$$x_{n+1} = rx_n + k. \quad (9b)$$

If a limit cycle exists, then $x_n \rightarrow x$ for large n , in which case (9b) yields

$$x = \frac{k}{1-r}. \quad (10)$$

⁴ This impulse approximation, and the resultant phase diagram, is discussed in [1]. Note, however, in [1] it is stated that $k_- > k_+$, which is not the case.

Consider now how this limit is approached. From (9b)

$$\begin{aligned}
 x_1 &= rx_0 + k \\
 x_2 &= r^2x_0 + (1+r)k \\
 x_3 &= r^3x_0 + (1+r+r^2)k \\
 &\vdots \\
 x_n &= r^n x_0 + k \sum_{j=0}^{n-1} r^j + \varepsilon r^{n-m}.
 \end{aligned} \tag{11}$$

Here we have assumed that a small disturbance of magnitude ε has influenced the pendulum at time t_m , $m < n$. We see that, in the limit of large n , $x_n \rightarrow x$. Thus the pendulum angular speed becomes independent of the initial value x_0 , and the limit cycle is stable against irregular disturbances ε . The condition for this stability is $k > 0$, i.e. $k_+ > k_-$ from (10), since x must be positive. We note also from (10) that for small friction (such that $\frac{1}{2}b\tau \ll 1$) the maximum angular speed is

$$x \approx \frac{k\omega}{\pi b}. \tag{12}$$

Thus we have shown that the action of a pendulum clock escapement results in a stable limit cycle. The method we have used is most suitable to this problem because of the impulsive forces. The more usual method for establishing the existence of limit cycles is different [10, 11]. Due originally to Poincaré, it is based upon the idea of tangent circles. The displacement and angular velocity are represented in polar co-ordinates ($\theta \sim R \sin(\phi)$, $\dot{\theta} \sim R \cos(\phi)$). Substituting these into the equation of motion, equation (13) below, we seek a minimum and maximum radius for the phase plot. If these radii exist, then a limit cycle exists. This approach is better adapted to systems with continuous forces, but it can be applied to the present case, with care, and yields the same result: a stable limit cycle with maximum angular speed given by equation (12).

5. Solution to the equation of motion

The linearized equation of the pendulum with escapement is

$$\ddot{\theta} + b\dot{\theta} + \omega_0^2\theta \approx \frac{1}{\Delta t} p(\theta, \dot{\theta}). \tag{13}$$

We expect small pendulum amplitudes for grandfather clocks, $<5^\circ$, and so the linear approximation is a very good one. $p(\theta, \dot{\theta})$ represents the (angular) momentum transferred to the pendulum by the escapement mechanism, during the short time interval Δt . It can be written as [1]

$$p(\theta, \dot{\theta}) = \begin{cases} k_+\delta(\theta), & \dot{\theta} > 0 \\ k_-\delta(\theta), & \dot{\theta} < 0 \end{cases} \tag{14}$$

for the recoil escapement, where

$$\delta(\theta) = \begin{cases} 1 & \text{if } |t - 2n\pi/\omega| < \frac{1}{2}\Delta t \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

This assumes that the escapement influence is a series of small impulses every half cycle. These impulses impart angular acceleration $\alpha_{\pm} = k_{\pm}/\Delta t$.

The action of the escapement can be represented by Green functions as follows. Write for the solution of equation (13):

$$\theta(t) = \sum_{n=0}^{\infty} G(t - t_n)k_n + \text{transients.} \tag{16}$$

Reference [9] where

$$k_n = \begin{cases} k_+, & n \text{ even} \\ k_-, & n \text{ odd} \end{cases} \quad (17)$$

and where Green functions for the damped oscillator are given by

$$G(t - t_n) = \begin{cases} \frac{1}{\omega} \exp\left(-\frac{1}{2}b(t - t_n)\right) \sin(\omega(t - t_n)), & t \geq t_n \\ 0, & t < t_n. \end{cases} \quad (18)$$

This solution has the form required for the damped pendulum (equation (8)) with impulses $k_n = k_{\pm}$ added to the angular velocity $\dot{\theta}$ at time t_n . This follows, since from equation (15) $\dot{G}(t = t_n) = 1$ and so from (16) $\dot{\theta}(t) = \sum_{n=0}^m k_n$ for $t > t_m$. Thus impulses k_n are added at t_n , as in figure 3 and equation (14). $G(t - t_n)$ represents the response of the pendulum to a unit impulse at time t_n .

It is common in physics and frequently convenient in dynamical systems analysis to represent cyclic forces by spectral decomposition, particularly if a single frequency dominates. For such sudden blows as we have here, however, representation by a Fourier series is not very satisfactory because there are many frequencies which contribute. The physics of clock escapements is an excellent vehicle for demonstrating the efficacy of Green functions.

If a limit cycle exists, then we can calculate θ from (16):

$$\theta_L(t) = \lim_{N \rightarrow \infty} \sum_{n=0}^N G(t_N - t_n) k_n. \quad (19)$$

Substituting from (17) and (18) yields, after some calculation,

$$\theta_L(t) = \frac{\sin(\omega t)}{\omega} \frac{k_+ + k_- \exp(\frac{1}{4}b\tau)}{1 - \exp(-\frac{1}{2}b\tau)}. \quad (20)$$

Again we assume that friction is small, in which case

$$\theta_L \approx \frac{k}{\pi b} \sin(\omega t). \quad (21)$$

Thus an undamped oscillatory solution exists, with the same frequency as for the damped pendulum. The amplitude is independent of the initial amplitude. It is also independent of ω , as for the free pendulum. Note that the magnitude of angular velocity obtained by differentiating (21) is the same as that found earlier, equation (12).

6. Numerical integration

The equation of motion (13) is readily integrated. We plot in figure 4 the resulting phase diagram for a grossly exaggerated choice of parameters: $(b, k) = (0.22, 0.1 \text{ s}^{-1})$. We have chosen for simplicity a dead-beat escapement, with a single beat per cycle, so that the k_- impulse of equation (14) is missing. During the integration, angular momentum p of equation (14) is added to $\dot{\theta}$ at $\theta = 0$, or else force $p/\Delta t$ is added to $\ddot{\theta}$; both yield the same result. Note that the system is stable: the phase trajectory is a spiral into a limit cycle. k_+ has been chosen to be very large so that the impulse at $\theta = 0$ is distinct. From equation (21) the limit cycle amplitude is calculated as 8.3° , which is close to that obtained numerically. Note the slight asymmetry. Due to the impulse, peak amplitude is greater for positive θ (at 9.1°) than for negative θ (at 8.2°).

This confirms the analysis of sections 4 and 5. Further numerical calculations show that stability also obtains for larger amplitudes, where the linear approximation of equation (13) is not valid, though in this case the limit cycle amplitude of equation (21) underestimates the true value.

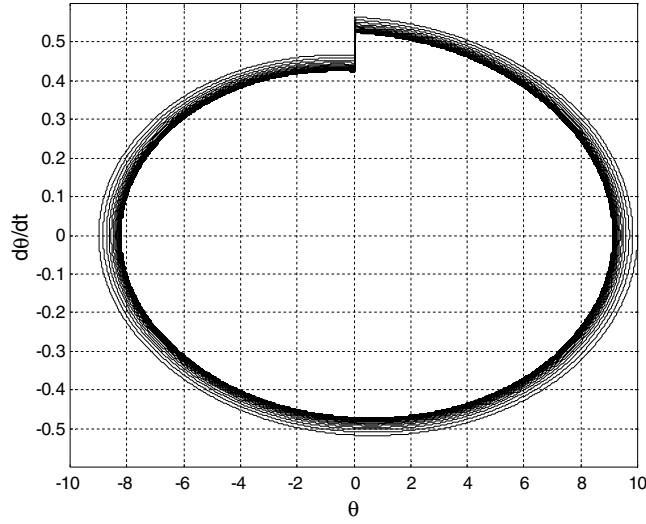


Figure 4. Phase diagram for a numerical solution of equation (13), assuming a single impulse per cycle (dead-beat escapement). The parameter values are: $\ell = 1$ m, $b = 0.22$, $k = 0.1$ s⁻¹. Initial values are $(\theta, \dot{\theta}) = (10^\circ, 0^\circ)$. Note how the trajectory spirals into a limit cycle.

7. Energy loss

Equation (13) is of the form

$$\ddot{\theta} + h(\theta, \dot{\theta}) + y(\theta) = 0 \quad (22)$$

with

$$h(\theta, \dot{\theta}) = b\dot{\theta} - \frac{1}{2}\alpha \operatorname{sgn}(\dot{\theta})\delta(\theta) \quad (23)$$

where $\alpha = k/\Delta t$ is angular acceleration. The rate at which external energy is supplied to a system described by (22) is given by [1]

$$\dot{E} = -\dot{\theta}h(\theta, \dot{\theta}) = -b\dot{\theta}^2 + \frac{1}{2}\alpha|\dot{\theta}|\delta(\theta). \quad (24)$$

Thus, energy is lost by the pendulum except at $\theta = 0$, as we expect. (Recall that stability requires $k > 0$, so that energy is pumped into the pendulum at $\theta = 0$.)

We can estimate from energy considerations the system parameters k and b in terms of macroscopic parameters, as follows. Consider the system of figure 4, with only one impulse k per cycle. The energy gained by the pendulum per cycle is $\delta E = m\ell^2 k^2$, whereas the energy imparted to the pendulum by the descending weight is $\delta E = \varepsilon mgh/N$ per cycle. Here ε is an efficiency factor, which depends upon escapement details and gear train efficiency; $\varepsilon = 25\%$ is a typical and realistic value. N is the number of cycles powered by the weight as it descends a height h between windings. We see that this is estimated to be $N = \varepsilon gh/(\ell k)^2$. The longcase clock operates for an interval $N\tau$ between windings, where $\tau = 2\pi\sqrt{\ell/g}$ is the pendulum period. From these considerations, we can express the escapement parameter k as

$$k = \sqrt{\frac{2\pi\varepsilon h}{N\tau} \frac{g}{\ell}}. \quad (25)$$

Choosing $N\tau = 1$ week and $h = \ell = 1$ m yields $k \approx 0.003$ s⁻¹. From (21) the peak amplitude $\theta_0 = k/\pi b$ and so $\theta_0 = 3^\circ$ yields $b \approx 0.019$ s⁻¹. Thus, the requirement that longcase clocks beat at 1 s intervals (period 2 s, hence determining $\ell = 1$ m), and the

requirement that they need winding only once per week, impose a stringent constraint upon the maximum friction and escapement parameter values. Noting that some longcase clocks required winding only once per year, and we can readily appreciate the precision engineering skills of the manufacturers.

8. Conclusion and discussion

Weight-driven pendulum clocks provide an interesting, practical, and historically important dynamical system calculation for the student. The natural tool to provide us with a solution to this problem is Green functions. The solution obtained yields pendulum oscillation amplitude in terms of the two system parameters: pendulum friction coefficient b and escapement impulse parameter k . Clearly, it is practically desirable to reduce friction, since this will reduce wear and increase clock longevity. Yet friction is an essential element of the self-regulation. A profusion of escapement mechanisms were developed in the 200 years following the first anchor escapement (well described and illustrated in, e.g., [12]), each with its own characteristic k_{\pm} and b . These had differing efficiencies, wear characteristics and actions. The ingenious ‘pinwheel’ escapement was popular in France but criticized in England for requiring frequent lubrication (this indicates the dichotomy of the problem, illustrated in our analysis, of wanting friction but only a little). The amazing ‘grasshopper’ escapement of Harrison was efficient and required no maintenance: the wood of which it was made provided sufficient oils for lubrication, and one that he built still functions today.

Understanding something of the physics of these devices teaches students about dynamical systems (stability, limit cycles) and provides a practical application of Green functions. For all physicists, it enhances our appreciation of the ingenuity and skill of the early clockmakers.

Acknowledgment

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